Gelfand Pairs and Beyond - Exercises

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1. *Let G be a finite group. Then G is abelian if and only if all irreducible unitary representations of G are one-dimensional.*

Let G be a finite group and (π, V) be a representation of G. Since $\pi(g)\pi(g') = \pi(g')\pi(g)$ for all $g,\ g' \in G$, $\pi(g) = \lambda_g 1\,(\lambda_g \in \mathbb{C})$ by Schur's Lemma. Therefore, being as all subspaces of V are $\pi(G)$-invariant, the representation space V must be one-dimensional in order for π to be irreducible.

Conversely, let G be a finite group and all $\pi_i \in \widehat{G}$ $(1 \leq i \leq d)$ are one-dimensional. Then G is an abelian group. Indeed: by Burnside theorem, we have

$$|G| = \sum_{i=1}^{d} n_i^2$$

where d is number of conjugacy classes and each n_i $(1 \leq i \leq d)$ is dimension of the irreducible unitary representation of π_i. Since all $n_i = 1$, $d = |G|$ and every conjugacy class consists of one element. Take $x \in G_i$ then $yxy^{-1} = x$ for all $y \in G$. Hence G is abelian.

2. *Let G be a finite abelian group and H a subgroup of G. Show that every unitary character of H can be extended to a unitary character of G.*

Let $\widehat{G}$ be a character group of G and H be a subgroup of G. $H^* := \{\chi \in \widehat{G}\ ;\ \chi(h) = 1\ (\forall h \in H)\}$. It is only necessary to show that the natural map $f : \widehat{G} \ni \chi \mapsto \chi|_H \in \widehat{H}$ is surjective. Since $\operatorname{Ker} f = H^*$ and $H^* \simeq (G/H)^\wedge$, then

$$|\operatorname{Im} f| = \frac{|\widehat{G}|}{|\operatorname{Ker} f|} = \frac{|\widehat{G}|}{|(G/H)^\wedge|} = \frac{|\widehat{G}||\widehat{H}|}{|\widehat{G}|} = |\widehat{H}|.$$

Hence f is surjective.

3. *Prove the same as in exercise 1 for a compact group G.*

If G is a compact abelian group, what all irreducible unitary representation of G are one-dimensional is clear by Schur's lemma.

Conversely, let G be a compact group and all $\pi \in \widehat{G}$ are one-dimensional. Then G is a abelian group. Indeed: take $f \in L^2(G)$, by Peter-Weyl theorem, we have

$$f = \sum_{\chi \in \widehat{G}} (f,\, \chi)\chi$$

where $(f,\, \chi) = \int_G f(x)\overline{\chi(x)}dx$. Take $f,\ g \in L^2(G)$, then $f * g \in L^2(G)$ and since

$$\begin{aligned}
(f * g,\, \chi) &= \int_G (f * g)(x)\overline{\chi(x)}dx \\
&= \int_G \int_G f\left(xy^{-1}\right) g(y)\overline{\chi(x)}dydx \\
&= \int_G \int_G f\left(z\right) g(y)\overline{\chi(zy)}dydz \\
&= \int_G \int_G f\left(z\right) g(y)\overline{\chi(z)\chi(y)}dydz \quad (\because \pi \text{ is one-dimensional.}) \\
&= \int_G f\left(z\right) \overline{\chi(z)}dz \int_G g(y)\overline{\chi(y)}dy \\
&= (f,\, \chi)(g,\, \chi)
\end{aligned}$$

for all $x \in G$, we have

$$f * g = \sum_{\chi \in \widehat{G}} (f * g,\, \chi)\chi = \sum_{\chi \in \widehat{G}} (f,\, \chi)(g,\, \chi)\chi.$$

So $f * g = g * f$, that is $\int_G f(yx^{-1})g(x)dx = \int_G g(yx^{-1})f(x)dx$. Since $\int_G g(yx^{-1})f(x)dx = \int_G g(x)f(x^{-1}y)dx$, then $f(yx) = f(xy)$ for all $x,\ y \in G$. So $yx = xy$ for all $x,\ y \in G$. Hence G is abelian.

4. *Determine (up to equivalence) the irreducible unitary representations of the groups $SO(2)$ and $O(2)$.*

6. *Let Φ be the unitary representation of $SL(2, \mathbb{R})$ on $V = L^2(\mathbb{R}^2)$ given by $\Phi(g)f(x) = f(g^{-1}x)$.*

(a) *Show that for each $r > 0$ that the operator $L_r : V \to V$ given by $L_r f(x) = f(rx)$ commutes with $\Phi(g)$ for all $g \in SL(2, \mathbb{R})$.*

Let $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\, \mathbb{R})$, then $g^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

$$\begin{aligned}
\left(L_r\left(\Phi(g)f\right)\right)\begin{bmatrix} x \\ y \end{bmatrix} = \left(\Phi(g)f\right)\begin{bmatrix} rx \\ ry \end{bmatrix} = f\left(g^{-1}\begin{bmatrix} rx \\ ry \end{bmatrix}\right) = f\begin{bmatrix} rdx - rby \\ -rcx + ray \end{bmatrix}, \\
\left(\Phi(g)\left(L_r f\right)\right)\begin{bmatrix} x \\ y \end{bmatrix} = \left(L_r f\right)\left(g^{-1}\begin{bmatrix} x \\ y \end{bmatrix}\right) = \left(L_r f\right)\begin{bmatrix} dx - by \\ -cx + ay \end{bmatrix} = f\begin{bmatrix} rdx - rby \\ -rcx + ray \end{bmatrix}.
\end{aligned}$$

$$\begin{array}{ccc}
V & \xrightarrow{\Phi(g)} & V \\
{\scriptstyle L_r}\big\downarrow & & \big\downarrow{\scriptstyle L_r} \\
V & \xrightarrow{\Phi(g)} & V
\end{array}$$

Hence we have $L_r(\Phi(g)f) = \Phi(g)(L_r f)$ for all $g \in SL(2,\, \mathbb{R})$.

(b) *Conclude that Φ is reducible.*

Assume Φ is irreducible. Since L_r commutes with $\Phi(g)$, L_r is a scalar operator by Schur's Lemma. But now L_r is not a scalar operator. Hence this is contradiction.

7. *Show that the pair $(SU(2), SO(2))$ is a compact Gelfand pair.*

First, show that $SU(2)$ is decomposed as

$$SU(2) = SO(2)TSO(2) \tag{$*$}$$

by using the maximal torus $T := \{t(\varphi) = \mathrm{diag}(e^{i\varphi/2}, e^{-i\varphi/2})\,;\, \varphi \in \mathbb{R}\}$ of $SU(2)$. Set $g = \begin{bmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{bmatrix} \in SU(2)$, $\alpha = a_1 + ia_2$, $\beta = b_1 + ib_2$ and $k(\tau) = \begin{bmatrix} \cos(\tau/2) & -\sin(\tau/2) \\ \sin(\tau/2) & \cos(\tau/2) \end{bmatrix} \in SO(2)$. Since $a_1^2 + a_2^2 + b_1^2 + b_2^2 = 1$, take $\varphi \in \mathbb{R}$ such that

$$\sqrt{a_1^2 + b_1^2} = \cos\frac{\varphi}{2},\ \sqrt{a_2^2 + b_2^2} = \sin\frac{\varphi}{2}.$$

And so take $\tau_1,\ \tau_2 \in \mathbb{R}$ such that

$$a_1 = \cos\frac{\varphi}{2}\cos\tau_1,\ b_1 = \cos\frac{\varphi}{2}\sin\tau_1,\ a_2 = \sin\frac{\varphi}{2}\cos\tau_2,\ b_2 = \sin\frac{\varphi}{2}\sin\tau_2.$$

Set $\tau = \tau_1 + \tau_2$, $\tau' = \tau_1 - \tau_2$, then

$$\begin{aligned}
k(\tau)t(\varphi)k(\tau') &= \begin{bmatrix} \cos\frac{\tau}{2} & -\sin\frac{\tau}{2} \\ \sin\frac{\tau}{2} & \cos\frac{\tau}{2} \end{bmatrix} \begin{bmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{bmatrix} \begin{bmatrix} \cos\frac{\tau'}{2} & -\sin\frac{\tau'}{2} \\ \sin\frac{\tau'}{2} & \cos\frac{\tau'}{2} \end{bmatrix} \\
&= \begin{bmatrix} e^{i\varphi/2}\cos\frac{\tau}{2} & -e^{i\varphi/2}\sin\frac{\tau}{2} \\ e^{i\varphi/2}\sin\frac{\tau}{2} & e^{i\varphi/2}\cos\frac{\tau}{2} \end{bmatrix} \begin{bmatrix} \cos\frac{\tau'}{2} & -\sin\frac{\tau'}{2} \\ \sin\frac{\tau'}{2} & \cos\frac{\tau'}{2} \end{bmatrix} \\
&= \begin{bmatrix} e^{i\varphi/2}\cos\frac{\tau}{2}\cos\frac{\tau'}{2} - e^{i\varphi/2}\sin\frac{\tau}{2}\sin\frac{\tau'}{2} & -e^{i\varphi/2}\cos\frac{\tau}{2}\sin\frac{\tau'}{2} - e^{i\varphi/2}\sin\frac{\tau}{2}\cos\frac{\tau'}{2} \\ e^{i\varphi/2}\sin\frac{\tau}{2}\cos\frac{\tau'}{2} + e^{i\varphi/2}\cos\frac{\tau}{2}\sin\frac{\tau'}{2} & -e^{i\varphi/2}\sin\frac{\tau}{2}\sin\frac{\tau'}{2} + e^{i\varphi/2}\cos\frac{\tau}{2}\cos\frac{\tau'}{2} \end{bmatrix} \\
&= \begin{bmatrix} \cos\frac{\varphi}{2}\cos\frac{\tau+\tau'}{2} + i\sin\frac{\varphi}{2}\cos\frac{\tau-\tau'}{2} & -\cos\frac{\varphi}{2}\sin\frac{\tau+\tau'}{2} + i\sin\frac{\varphi}{2}\sin\frac{\tau-\tau'}{2} \\ \cos\frac{\varphi}{2}\sin\frac{\tau+\tau'}{2} + i\sin\frac{\varphi}{2}\sin\frac{\tau-\tau'}{2} & \cos\frac{\varphi}{2}\cos\frac{\tau'+\tau}{2} - i\sin\frac{\varphi}{2}\cos\frac{\tau'-\tau}{2} \end{bmatrix} \\
&= \begin{bmatrix} a_1 + ia_2 & -b_1 + ib_2 \\ b_1 + ib_2 & a_1 - ia_2 \end{bmatrix} \\
&= \begin{bmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{bmatrix} \\
&= g
\end{aligned}$$

Hence we have equation $(*)$.

Next, consider the involution $\theta : SU(2) \to SU(2)$ defined by

$$\theta(g) := \overline{g}\ \ (g \in SU(2)).$$

Clearly θ leaves $SO(2)$ fixed and $\theta(t(\varphi)) = \overline{t(\varphi)} = t(\varphi)^{-1}$ for all $t(\varphi) \in T$. Therefore, because $SU(2) = SO(2)TSO(2)$, we have

$$\theta(g) \in SO(2)g^{-1}SO(2)$$

for all $g \in SU(2)$. So we have by Proposition 7.3: the pair $(SU(2), SO(2))$ is a Gelfand pair.

8. *Show that the pairs* $(U(n), U(1) \times U(n-1))$ *are compact Gelfand pairs for all* $n = 1, 2, 3\ldots$